Stationary solution for the color-driven Duffing oscillator

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Adiabatic methods may be used to find approximate reduced stationary distributions for the linearly damped Duffing oscillator driven by correlated noise of finite bandwidth. Here we show that a stationary distribution may be obtained which explains the insensitivity of the system to variations in damping coefficient and noise color as observed in numerical simulations. Exact analytic results are obtained for parabolic potentials for comparison. [S1063-651X(96)01306-2]

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The macroscopic behavior of many physical, chemical, and biological processes may be understood in terms of reduced models in just one or a few relevant variables. In nonlinear dynamical systems exhibiting transitions from regular to chaotic motion, such reduced models have been central to allowing tractable analyses to be performed near the critical region. A case in point is the Duffing oscillator which has been used in the description of a number of phenomena, including Rayleigh-Bénard convection [1,2], bifurcations in certain nonlinear electrical oscillators [3], and vibrations of buckled mechanical beams [4].

Recently the Duffing oscillator driven by a stochastic forcing term of finite bandwidth has been of interest [5–9]. It has been observed that the system is insensitive to the precise value of either the damping coefficient or the noise color. Indeed, the approximate analytic expression for the reduced stationary distribution of Wu, Billah, and Shinozuka [9] appears to be independent of either of these parameters and yet is in excellent agreement with numerical simulations. We show that the adiabatic elimination method does, in fact, yield a stationary distribution that depends explicitly on both the damping coefficient and the noise color and demonstrate the conditions under which there is no sensitivity to the exact numerical values of these parameters.

We consider the system

$$\ddot{x}+\lambda\dot{x}+U'(x)=\eta(t), \tag{1a}$$

$$\langle\eta(t)\eta(0)\rangle=D\gamma e^{-\gamma|t|}, \tag{1b}$$

where λ is a damping coefficient, $U(x)=\frac{1}{2}ax^2+\frac{1}{4}bx^4$ $(b>0)$ is the Duffing potential, and $\eta(t)$ is an exponentially correlated (Ornstein-Uhlenbeck) noise process with strength D and bandwidth γ. The system is different from that considered in stabilization problems [10] in that the noise here is additive not multiplicative.

Putting $\dot{x}=v$, Eq. (1) admits the Fokker-Planck equation for the distribution $p(x,v,\eta,t)$,

$$\frac{\partial}{\partial t}p=(L_a+L_b+L_i)p, \tag{2a}$$

$$L_a=-\frac{\partial}{\partial x}v+\frac{\partial}{\partial v}[\lambda v+U'(x)], \tag{2b}$$

$$L_b=\gamma\frac{\partial}{\partial\eta}\left(\eta+D\gamma\frac{\partial}{\partial\eta}\right),\quad L_i=-\frac{\partial}{\partial v}\eta. \tag{2c}$$

We wish to find the reduced stationary distribution $\sigma_0(x)=\lim_{t\to\infty}\sigma(x,t)$, where $\sigma(x,t)=\int dv\,d\eta\;p(x,v,\eta,t)$. For this we require the evolution equation for $\sigma(x,t)$, i.e., we want to reduce Eq. (2) to the form $\partial_t\sigma=L_\sigma\sigma$. We do this in two steps using the methods of Refs. [11] and [12] by first eliminating the noise variable η from the above Fokker-Planck equation, followed by the velocity variable v.

The presence of $U'(x)$ in the expression for L_a precludes an exact elimination of η. However, since we are primarily interested in obtaining $\sigma_0(x)$ we may eliminate η in the long-time limit as an expansion in inverse powers of the noise bandwidth γ. Denoting as L_r the reduced operator obtained after elimination of η for $t\to\infty$, we have the exact formal expression [11]

$$L_r=L_a+D\sum_{n=0}^{\infty}\frac{1}{\gamma^n}\partial_v(L_a^{\times})^n\partial_v\,, \tag{3}$$

where $L_a^{\times}\partial_v=[L_a,\partial_v]$. The lowest two commutators in this expansion are

$$L_a^{\times}\partial_v=\partial_x-\lambda\partial_v\,, \tag{4a}$$

$$(L_a^{\times})^2\partial_v=-\lambda\partial_x+[\lambda^2-U''(x)]\partial_v\,. \tag{4b}$$

These are sufficient to obtain the lowest-order corrections to the stationary distribution due to the noise color, assuming $\gamma\to\infty$, $\gamma\gg\lambda$.

It turns out that the $U''(x)$ term in Eq. (4) hinders the elimination of the velocity v from L_r. To simplify the problem and allow some progress, we replace $U''(x)$ by the mean value $\langle U''\rangle$ it yields during the evolution of x (a "mean-field" approximation [13]). Then, keeping terms up to $n=2$ in Eq. (3), we may write $L_r=L_1+L_v$ with

$$L_1=-\frac{\partial}{\partial x}v+\frac{\partial}{\partial v}g(x),\quad L_v=\lambda\frac{\partial}{\partial v}\left(v+Q\frac{\partial}{\partial v}\right), \tag{5a}$$

$$g(x)=U'(x)+\frac{D}{\lambda}\left(\frac{\lambda}{\gamma}-\frac{\lambda^2}{\gamma^2}\right)\frac{\partial}{\partial x}, \tag{5b}$$

$$Q=\frac{D}{\lambda}\left(1-\frac{\lambda}{\gamma}+\frac{\lambda^2}{\gamma^2}-\frac{\langle U''\rangle}{\gamma^2}\right). \tag{5c}$$

If $\langle U''\rangle$ is replaced by $U''(x)$ these expressions are correct to $O(\gamma^{-2})$. Inclusion of higher commutators results in considerable complications and makes the elimination of v much harder.

Using $\mathcal{U}=e^{v^2/4Q}$ we now make the transformation $H_r=-\mathcal{U}L_r\mathcal{U}^{-1}$, which we may write as

$$H_r=\lambda B^\dagger B+\overline{A}\overline{B}+AB^\dagger, \tag{6a}$$

$$B=\frac{v}{2\sqrt{Q}}+\sqrt{Q}\frac{\partial}{\partial v},\quad B^\dagger=\frac{v}{2\sqrt{Q}}-\sqrt{Q}\frac{\partial}{\partial v}, \tag{6b}$$

$$A=\frac{1}{\sqrt{Q}}\left(g(x)+Q\frac{\partial}{\partial x}\right),\quad \overline{A}=\sqrt{Q}\frac{\partial}{\partial x}. \tag{6c}$$

Noting that the commutator of A and $\overline{A}$ is $[\overline{A},A]=U''(x)$, we see that the system represented by Eq. (6) is analogous to that of Brownian motion in an external field [14,15].

The most rapid method of effecting the elimination of v from H_r is to use the stochastic version of the Rayleigh-Schrödinger expansion developed in Ref. [12]. This gives directly the reduced operator in powers of the inverse damping coefficient:

$$L_\sigma=\frac{1}{\lambda}\overline{A}A+\frac{1}{\lambda^3}\overline{A[\overline{A},A]}A+O(\lambda^{-5}). \tag{7}$$

The first term in this yields the $O(\lambda^{-2})$ equation

$$\frac{\partial}{\partial t}\sigma(x,t)=\frac{1}{\lambda}\frac{\partial}{\partial x}\left(U'(x)+\frac{D^*}{\lambda}\frac{\partial}{\partial x}\right)\sigma(x,t), \tag{8a}$$

$$D^*=D\left(1-\frac{\langle U''\rangle}{\gamma^2}\right). \tag{8b}$$

The corresponding stationary distribution is

$$\sigma_0(x)=N\exp\left(-\lambda\frac{U(x)}{D^*}\right), \tag{9}$$

where N is a normalization coefficient. The form of $\sigma_0(x)$ is identical to that found by others [9] except for the explicit appearance of the damping and noise-color parameters.

We may readily verify the correctness of the above stationary distribution in the white-noise limit $\gamma\to\infty$ by solving the Duffing oscillator problem Eq. (1a) with a white-noise forcing term, $\langle\eta(t)\eta(0)\rangle=2D\delta(t)$. The resulting Fokker-Planck operator

$$L=-\frac{\partial}{\partial x}v+\frac{\partial}{\partial v}U'(x)+\lambda\frac{\partial}{\partial v}\left(v+\frac{D}{\lambda}\frac{\partial}{\partial v}\right), \tag{10}$$

corresponds to that of the Kramers-Klein equation of Brownian motion theory [16]. The adiabatic elimination of the velocity variable from it results in the Smoluchowski equation to lowest order [14–16], which is exactly Eq. (8) with D^* replaced by D. In other words, the limit $\gamma\to\infty$ is correctly reproduced by Eq. (9).

This result explains the lack of noise-color dependence in analyses of the Duffing oscillator power spectrum [5,6]. The effect of color is only apparent at order γ^{-2} in the bandwidth, which in the limit $\gamma\to\infty$ is a small correction. The analysis also shows that to order γ^{-2} we may replace the colored-noise source in the Duffing equation (1a) by a white-noise source of noise strength D^*. This color-modified, or effective, noise strength contains only contracted information of the original colored-noise process but simplifies considerably the handling of the Duffing oscillator problem.

The above expression for $\sigma_0(x)$ further explains the excellent agreement found by Wu, Billah, and Shinozuka [9] between their approximate stationary distribution, which was independent of the damping coefficient (and of the noise color γ), and their numerical simulations. In the large-damping limit $\lambda\to\infty$ the normalization coefficient N in Eq. (9) may be evaluated by the method of steepest descents [17,18]. Since the Duffing potential is symmetric, $U(-x)=U(x)$, it has two minima at $x=\pm\sqrt{|a|/b}$ for $a<0$ and just the single minimum at $x=0$ for $a>0$ (we ignore potentials for which $a=0$). Denoting $\mu=1$ or 2 as the number of minima of $U(x)$ we obtain the asymptotic result

$$\sigma_0(x)\sim\left(\frac{\lambda U_0''}{2\pi D^*\mu^2}\right)^{1/2}e^{-\lambda[U(x)-U_0]/D^*}, \tag{11}$$

where (U_0,U_0'') is $(-a^2/4b,2|a|)$ for $a<0$ and $(0,a)$ for $a>0$. From this,

$$\frac{\partial\sigma_0}{\partial\lambda}\propto\left(\frac{1}{2\sqrt{\lambda}}-\frac{\sqrt{\lambda}[U(x)-U_0]}{D^*}\right)e^{-\lambda[U(x)-U_0]/D^*}, \tag{12}$$

with the consequence that

$$\lim_{\lambda\to\infty}\frac{\partial\sigma_0}{\partial\lambda}=0. \tag{13}$$

In other words, for large damping the stationary distribution $\sigma_0(x)$ becomes independent of λ altogether and the dynamics of the oscillator becomes concentrated near its attracting fixed points.

The stationary moments

$$\langle x^n\rangle=\int_{-\infty}^{\infty}dx\,x^n\sigma_0(x) \tag{14}$$

may also be evaluated in the asymptotic limit $\lambda\to\infty$ by steepest descents [18]. When $a<0$ we find

$$\langle x^n\rangle=(|a|/b)^{n/2} \tag{15}$$

for n even and $\langle x^n\rangle=0$ for n odd. This approximation is good when the two minima are as far apart as possible, i.e., $|a|\gg b$. When $a>0$,

$$\langle x^n \rangle = \frac{1}{\sqrt{\pi}} \left(\frac{2D^*}{\lambda a} \right)^{n/2} \Gamma\left(\frac{n+1}{2} \right) \tag{16}$$

for n even and again $\langle x^n \rangle = 0$ for n odd. In particular, the second moment is

$$\langle x^2 \rangle = D^*/\lambda a. \tag{17}$$

The above asymptotic results should work best when $\gamma \gg \lambda$, $\langle U'' \rangle \ll \gamma^2$, and λ is large.

Notice that Eqs. (16) and (17) are the exact moments if $a > 0$, $b = 0$ given in Eq. (9). In fact, for $b = 0$ we may effect an exact reduction of the full Fokker-Planck equation (2) to give exact expressions for both the stationary distribution and the moments, which we may use to test the asymptotic results above. Since the problem is then in effect a linear Ornstein-Uhlenbeck process it should, in principle, be solvable by other methods of adiabatic elimination. When $U(x) = \frac{1}{2}ax^2$ $(a > 0)$, we have

$$L_a = -\partial_x v + \partial_v(\lambda v + ax). \tag{18}$$

We may readily show by induction that the commutators in Eq. (3) may be expressed entirely in terms of ∂_x and ∂_v. Thus, putting

$$(L_a^{\times})^n \partial_v = c_n \partial_x + g_n \partial_v, \tag{19}$$

we find

$$(L_a^{\times})^{n+1} \partial_v = g_n \partial_x - (\lambda g_n + a c_n) \partial_v. \tag{20}$$

This is true for $n = 0,1$ and hence for all n. Equations (19) and (20) yield the linear difference equation

$$g_{n+2} + \lambda g_{n+1} + a g_n = 0, \tag{21}$$

with $g_0 = 1$, $g_1 = -\lambda$, and the c_n sequence $c_{n+1} = g_n$, with $c_0 = 0$. Beyond the first two terms we have

$$g_2 = \lambda^2 - a, \quad g_3 = -\lambda^3 + 2\lambda a, \quad g_4 = \lambda^4 - 3\lambda^2 a + a^2, \tag{22}$$

and so on.

A closed expression for g_n may be obtained, for example, by using the z transform [19]

$$\phi(z) = \sum_{n=0}^{\infty} g_n z^{-n}, \tag{23}$$

for z complex. Multiplying Eq. (21) by z^{-n} and summing over n gives

$$\phi(z) = \frac{z^2}{z^2 + \lambda z + a}. \tag{24}$$

This may be shown to ''generate'' the sequence (22) by dividing top and bottom by z^2 and expanding $\phi(z)$ as a series in inverse powers of z.

The inverse z transform is

$$g_n = \frac{1}{2\pi i} \oint dz\, \phi(z) z^{n-1}, \tag{25}$$

where the contour encloses all singularities of $\phi(z)$. From Eq. (24), $\phi(z)$ has only the two simple poles at

$$z_{\pm} = -\frac{\lambda}{2} \pm \frac{1}{2}\sqrt{\lambda^2 - 4a}, \tag{26}$$

which using $\cosh w = -\lambda/2\sqrt{a}$ we may write conveniently in the form

$$z_+ = \sqrt{a} e^{w}, \quad z_- = \sqrt{a} e^{-w}. \tag{27}$$

Evaluating the corresponding residues then gives

$$g_n = \frac{\sinh w(n+1)}{\sinh w} a^{n/2}. \tag{28}$$

Some algebraic manipulation shows that this formula correctly reproduces the sequence (22).

Using Eq. (3) and the sequence g_n, the reduced Fokker-Planck operator L_r for the parabolic potential may be written as in Eq. (5a) with

$$g(x) = ax + D \sum_{n=0}^{\infty} \frac{g_n}{\gamma^{n+1}} \frac{\partial}{\partial x}, \quad Q = \frac{D}{\lambda} \sum_{n=0}^{\infty} \frac{g_n}{\gamma^n}. \tag{29}$$

The transformation leading to Eq. (6) gives

$$A = \frac{1}{\sqrt{Q}} \left(ax + D^* \frac{\partial}{\partial x} \right), \quad \overline{A} = \sqrt{Q} \frac{\partial}{\partial x}, \tag{30a}$$

$$D^* = D\left(1 - a \sum_{n=0}^{\infty} \frac{g_n}{\gamma^{n+2}} \right), \tag{30b}$$

where the recurrence relation (21) has been used to write D^*. The operators A and $\overline{A}$, with commutator $[\overline{A}, A] = a$, effectively define Brownian motion in a parabolic potential. Titulaer [15] has shown that an exact elimination of the velocity variable is possible for such a system. Using this result we find

$$L_\sigma = \frac{1}{\lambda_h} \frac{\partial}{\partial x} \left(ax + \frac{D^*}{\lambda} \frac{\partial}{\partial x} \right), \tag{31}$$

where in our notation $\lambda_h = (\lambda + \sqrt{\lambda^2 - 4a})/2a$. The stationary distribution due to this L_σ has exactly the form in Eq. (9) but with an effective noise strength given by Eq. (30b).

Inserting the formula for g_n into Eq. (30b) and using the identity [20]

$$\sum_{n=1}^{\infty} \rho^n \sinh nw = \frac{\rho \sinh w}{1 - 2\rho \cosh w + \rho^2} \quad (\rho^2 < 1), \tag{32}$$

with $\rho = \sqrt{a}/\gamma$, gives the exact result

$$D^* = D\left(1 - \frac{a}{\gamma^2 + \gamma\lambda + a} \right). \tag{33}$$

With this D^* the exact moments $(a > 0, b = 0)$ have the form given in Eq. (16). In particular,

$$\langle x^2\rangle = \frac{D}{\lambda a}\left(1 - \frac{a}{\gamma^2 + \gamma\lambda + a}\right). \tag{34}$$

Note that this expression is valid for arbitrary γ and λ although the order in which we eliminated η and v makes sense only if $\gamma > \lambda$ (more precisely, we require that the condition $D^* > 0$ holds).

Wu, Billah, and Shinozuka [9] have evaluated $\langle x^2\rangle$ numerically for the parabolic potential. We may compare their results to the exact values obtained from Eq. (34). Identifying the parameters used here to those of Ref. 9 (denoting these with an overbar) we have $\gamma = 1/\bar{\tau}$, $\lambda = \bar{\gamma}$, $a = \bar{\gamma}\bar{d}$, and $D = \frac{1}{2}\bar{Q}\bar{\gamma}^2$. Two sets of values were used by the above authors: (i) $\bar{\tau} = 0.01$, $\bar{\gamma} = 100$, $\bar{d} = \bar{Q} = 1$; and (ii) $\bar{\tau} = 0.01$, $\bar{\gamma} = 100$, $\bar{d} = 20.25$, $\bar{Q} = 0.5$. For these parameter values Eq. (34) gives $\langle x^2\rangle = 0.498$ and $\langle x^2\rangle = 0.0112$ to three significant figures for (i) and (ii), respectively. These are identical to the values obtained by the numerical evaluation.

The asymptotic formula Eq. (17), which requires $\gamma \gg \lambda$, $a \ll \gamma^2$, and λ large, gives $\langle x^2\rangle = 0.495$ and $\langle x^2\rangle = 0.009\,85$, respectively, for the two sets of parameters above, where we have used $\langle U''\rangle \approx a$. Actually, in neither case (i) nor (ii) is the condition $\gamma \gg \lambda$ ($1/\bar{\tau} \gg \bar{\gamma}$) met and in (ii) the value for a is quite large compared to γ^2. Nevertheless, the agreement is reasonable. Of course, this agreement improves when the required conditions are met. For example, when $\gamma = 10$, $\lambda = 5$, and $a = D = 1$ we obtain 0.1987 and 0.1980 for the exact and asymptotic second moments respectively; whereas for $\gamma = 100$, $\lambda = 10$, and $a = D = 1$ the difference occurs only in the seventh decimal place. In this case we have effectively $\langle x^2\rangle = D/\lambda a$. Small values of b do not affect the approximation. For larger values of b the asymptotic condition requires larger values of λ to compensate. Note also that the value of the colored-noise strength D does not affect the approximation at all.

In summary, we have derived a reduced stationary distribution for the color-driven Duffing oscillator, which contains a dependence on both the damping coefficient and the noise color. We have found that for large damping and weak color the stationary distribution becomes insensitive to the values of either of these parameters. This explains the excellent agreement observed between analytic approximations and numerical simulations performed by others. For parabolic potentials we have obtained the exact analytic reduced stationary distribution, allowing approximations to be tested directly.

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$$U(n,z) = \frac{e^{-z^2/4}}{\Gamma(n+1/2)}\int_0^\infty ds\, s^{n-1/2} e^{-zs-s^2/2}$$

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